Musica Universalis: From the Lambdoma of Pythagoras to the Tonality Diamond of Harry Partch

Seldom does it occur to the modern musician to question the system in which he/she works. For most Western musicians, twelve-tone equal temperament is simply a fact of life, the undisputed foundation of our art. Although microtonal composers and theorists do posit their alternative approaches, rarely is the question asked: what originally compelled us toward a twelve-tone system? The answer is to be found in the intersection between music and philosophy. The number twelve is directly connected to astrology, the signs of the zodiac. Each tone was meant to represent a position of the sun in the heavens, a connection to *musica mundana* or *musica universalis*, the "music of the spheres," an idea we trace back to the philosopher who supposedly discovered the relationship between tone and number: Pythagoras of Samos (ca. 570—ca. 490 BCE). Pythagoras believed that number was the essence of the universe, and that understanding the numerical proportions of harmonics was the key to understanding the universe—of literally unifying our consciousness with the mind of God. To this end he discovered or devised mathematical concepts such as the *mensa pythagorica*, a table of ratios derived from the study of the monochord, also known as the Lambdoma. The Lambdoma is considered by modern Pythagorean researchers to be one of the cornerstones of his philosophy.

Harry Partch, pioneer of the modern microtonal movement, studied the art and philosophy of the ancient Greeks at length, tracing what he learned from Helmholtz's *On the Sensations of Tone* and Kathleen Schlesinger's writings on Greek music back to the original sources, and what he perceived to be the original Greek aesthetic. Though it is doubtful that Partch studied the Pythagorean Lambdoma at any length, or ever discovered the writings of neo-Pythagoreans such as Albert von Thimus or Hans Kayser, the configuration and underlying concept of Partch's Tonality Diamond bears so much similarity to the Lambdoma that it must be regarded as a further development and distillation of the original abstract idea into practical application and corporeal form. By examining this connection we trace a line from the modern work of a twentieth century American composer back to a mathematical construct at the heart of ancient Pythagorean (and hence Platonic) teaching and philosophy, analyzing likewise how an instrument such as the Diamond Marimba is incompletely understood without pressing how ancient philosophy manifests itself in a mathematical conception of microtonality.

The Lambdoma: Source of Mathematical Principle in Tonal Systems

Far more than a simple mathematical table, the Lambdoma is a grand philosophical idea, the expression of a universal truth, as explained by Joscelyn Godwin:

> [T]he Pythagorean Table, whatever its origin, is an incomparable aid to speculative music, as a means toward symbolic explanation and possible illumination concerning cosmic and metaphysical realities. The Table is an image

> of the Universe. If extended to mathematical infinity it would contain every rational fraction and integer. Each one of these, expressed by a numerator and a denominator, is the product of an intersection between an overtone and an undertone row, i.e., every tone occurs as a member or one row of each type. If each is taken to represent one of the beings in the Universe, this dual origin emphasizes each being's dependence on a primordial duality, the initial split of the Lambdoma. One might say that whenever the two forces of contraction and expansion meet and are held in some proportional balance, a being arises—and a tone is sounded. Every being is both number and tone; both quantity and quality; both existence and value. All have the same root: the originating 1/1 tone that represents God the Creator. (Godwin 1995, 179)

Understanding the philosophical implications of the Pythagorean table reveals that the source of all mathematical understanding of tonal relationships actually lies in the concept of "oneness" and the splitting of this oneness into a proportionate ratio, creating tone. First, we shall examine and analyze the Lambdoma, after which we will trace the study and evolution of the table into a configuration known as the Tonality Diamond (shown below), invented in the twentieth century by American composer Harry Partch. All these two grids may appear to have in common are the fact that they are composed of ratios; but with knowledge of both tables as musical realities, the second will be revealed as a distillation of the first. Tracing the line of descent from one matrix to the other reveals a unity of science, philosophy, and art that has fallen into obscurity; yet it remains a living heritage for those passionate enough to trace music theory to its true roots.

The Pythagorean Table (12 x 12 grid)

1/1	2/1	3/1	4/1	5/1	6/1	7/1	8/1	9/1	10/1	11/1	12/1
1/2	2/2	3/2	4/2	5/2	6/2	7/2	8/2	9/2	10/2	11/2	12/2
1/3	2/3	3/3	4/3	5/3	6/3	7/3	8/3	9/3	10/3	11/3	12/3
1/4	2/4	3/4	4/4	5/4	6/4	7/4	8/4	9/4	10/4	11/4	12/4
1/5	2/5	3/5	4/5	5/5	6/5	7/5	8/5	9/5	10/5	11/5	12/5
1/6	2/6	3/6	4/6	5/6	6/6	7/6	8/6	9/6	10/6	11/6	12/6
1/7	2/7	3/7	4/7	5/7	6/7	7/7	8/7	9/7	10/7	11/7	12/7
1/8	2/8	3/8	4/8	5/8	6/8	7/8	8/8	9/8	10/8	11/8	12/8
1/9	2/9	3/9	4/9	5/9	6/9	7/9	8/9	9/9	10/9	11/9	12/9
1/10	2/10	3/10	4/10	5/10	6/10	7/10	8/10	9/10	10/10	11/10	12/10
1/11	2/11	3/11	4/11	5/11	6/11	7/11	8/11	9/11	10/11	11/11	12/11
1/12	2/12	3/12	4/12	5/12	6/12	7/12	8/12	9/12	10/12	11/12	12/12

Many believe that even the mathematical principles bearing the stamp of Pythagoras were actually invented later by successive generations of disciples, and so the surviving mystical teachings are thought to have more historical validity. Two concepts are crucial to this discussion: first, the dictum “All is Number;” and second, “the music of the spheres,” also known as *musica universalis* or *musica mundana*, the idea that the motions of the planets create tones that harmonize with one another and that this heavenly harmony is the model for earthly harmony, a model which applies not only to art but to individual and societal life. What is essential to an understanding of this second concept is the relationship of number to tone.

For Pythagoras and his brethren, number was First Principle, superseding all else; and number was, and remains, tone. This is not just in the sense that tone is frequency, as in vibrations per second measurable in Hertz (Hz). Tone-number correspondence is made apparent by the study of the harmonic series, a natural aural phenomenon that defines all intervallic tonal relationships as numerical. One of the discoveries Pythagoras is traditionally credited with is the relationship between number and tone. In his Manual of Harmonics, the Pythagorean philosopher Nicomachus of Gerasa (ca. 60-ca. 120 CE) tells the first written account of the legend of the Harmonious Blacksmith, a story still told today about how Pythagoras accomplished this signature achievement:

> One day he was deep in thought and seriously considering whether it could be possible to devise some kind of instrumental aid for the ears which would be firm and unerring...while thus engaged, he walked by a smithy and, by divine chance, heard the hammers beating out iron on the anvil and giving off in combination sounds which were most harmonious with each other, except for one combination.

> He recognized in these sounds the consonance of the octave, the fifth and the fourth. But he perceived that the interval between the fourth and the fifth was dissonant in itself but was otherwise complementary to the greater of these two consonances. Elated, therefore, since it was as if his purpose was being divinely accomplished, he ran into the smithy and found by various experiments that the difference of sound arose from the weight of the hammers, but not from the force of the blows, nor from the shapes of the hammers, nor from the alteration of the iron being forged. (Levin 1994, 83)

Nicomachus goes on to describe in detail how Pythagoras went home and experimented with weights and various proportional lengths of string until he fully identified the mathematical relationships involved. This story was passed on practically unaltered by various other writers (Adrastus, Gaudentius, Censorinus, Iamblichus, Macrobius, Boethius, and others), in spite of incongruities in the later descriptions. "One has only to try duplicating the experiment detailed by Nicomachus in this chapter to discover that it does not work, that the results adduced by Nicomachus simply cannot be obtained" (Levin 1994, 87), translator Flora Levin comments in her 1994 edition of the work. The ongoing endurance of the tale attests to its simple effectiveness in explaining this monumental realization, which formed the foundation of all Western music theory as we know it. Prior to this discovery tonality was purely a matter of intuition, done by the ear; Pythagoras showed us that mathematical relationships and tonal relationships were one and the same.

The Tonality Diamond: Partch's Resurrection of the Harmonic Series

An attempt at revival of ancient tonal theory began in the nineteenth century in Germany, in the work of music philosophers such as Albert von Thimus (1806-1878) and Hans Kayser (1891-1964), but it is through the efforts of American composer Harry Partch (1901-1974), after his reading of German scientist Hermann von Helmholtz (1821-1894), that a true vision of mathematical tonality awakens. Partch qualifies as a somewhat enigmatic figure by virtue of his complete lack of academic credentials and his artistic contribution to Western culture that, while historic, remains far outside the mainstream. Partch remained committed to a unique vision through the course of his life, in spite of the extreme hardships that vision engendered. Being largely self-taught, Partch made attempts early on at a career in performance, but was too full of antagonizing questions to fit comfortably into the mainstream. Early exposure to the music of China and Native America may have influenced him insofar as how he heard music, but he also spent long periods of time in isolation as a young man, and much of mainstream music never seemed "in tune" to him.

Oddly enough, Partch was correct; Western music is actually not "in tune." Our music is governed by the rules of piano tuning, which requires a "tempering" of tuning in order to sound uniform. Using a system that equalizes the distances between all semitones may have its advantages, but it is not "in tune" in the scientific sense; all intervals on a piano, with the exception of the octaves, are slightly detuned from the harmonic series. Perfect fifths on a piano are slightly flat, and major thirds are slightly sharp. It is not possible to tune a piano to the harmonic series and have all intervals sound the same across all the keys. While Partch intuited this to some degree, he did not envision a viable alternative until discovering a certain book at the Sacramento Library in 1923, an English translation of von Helmholtz's *On the Sensations of Tone*.

To situate Helmholtz's contribution to the mathematical principles underscoring the harmonic system, which Partch would encounter, we have to understand his training and purpose. Helmholtz, a famous scientist and acoustician, studied medicine at Friedrich Wilhelm Medical Institute in Berlin (M.D., 1843) and learned piano. He served as Professor Extraordinary at the Berlin Academy of Fine Arts, Director of Königsberg's Psychological Institute (1849-55), Professor of Anatomy and Physiology at University of Bonn (1855-58) and University of Heidelberg (1858-71). He was also a Professor of Physics at University of Berlin in 1871, the first director of Physico-Technical Institute in Berlin (1888), and was ennobled in 1882. He published *Lehre von den Tonempfindungen als physiologische Grundlage für die Theorie der Musik* in 1863, translated by A. Ellis as *On the Sensations of Tone as a Physiological Basis for the Theory of Music* in 1875 in London, and published in a New York edition in 1948.

Helmholtz's stated objective was to reestablish the mathematical basis for musical harmony, and in doing so reunify the ancient disciplines. "The horizons of physics, philosophy, and art have of late been too widely separated," he states. "And, as a consequence, the language, the methods, and the aims of any one of these studies present a certain amount of difficulty for the student of any other of them; and possibly this is the principal cause why the problem here undertaken has not been long ago more thoroughly considered and advanced towards its solution" (Helmholtz 1930, 1).

Through Helmholtz, and his own subsequent readings of ancient Greek music theorists, Partch developed his tonal theory and philosophy, explained in his *Genesis of a Music: an Account of a Creative Work, its Roots and its Fulfillments*. Partch's discussions in *Genesis* do include Plato and Pythagoras, but there is no mention whatsoever of the Lambdoma—unsurprising, since the only existing written authority in his time, Albert von Thimus' *Harmonikale Symbolik des Alterthums*, had never been translated into English (and it remains so to this day, other than one passage translated by Joscelyn Godwin for his compendium *The Harmony of the Spheres*). One passage in *Genesis*, however, reveals some knowledge of sources other than Helmholtz:

> We might, perhaps, congratulate ourselves that we are not as likely to develop music-minded Platos and Aristotles to condemn us as 'dangerous to the state' because of our musical acts and views, that we are not as likely to exile modern Timotheuses for introducing a few new scale tones (or are we?), and that we do not use or misuse 'semitones' by Chinese-fashioned edicts. But this is a superficial view. On second thought we must wonder why men of the Plato-Aristotle-Ptolemy-Zarlino-Mersenne-General Thompson breadth of interest take no part in musical discussions (aside from the 'appreciation' and 'I-love-music' attitudes), why there is so much unanimity on the content of music at the same time that battles rage over its commas and exclamation marks—that is to say, its

> ‘interpretation’; why there are so few ‘searchers’ in music as compared with the social and natural sciences, and even with literature...The pressing need is for a realignment of music with rationality—with something for a realistic, philosophic, or scientific bent to operate upon. (Partch, 1974, 56)

This shows that Partch was at least familiar with the theoretical treatises of Gioseffo Zarlino (1517-1590) and Marin Mersenne (1588-1648), two very influential writers whose researches into tuning lay further pavement along the way to the adoption of equal temperament. Zarlino was a Venetian composer, member of the Franciscan Order, author of *Institutioni Harmoniche* (1558) and “the most celebrated music theorist of the mid-16th century” (Encyclopedia Britannica, 2013). Mersenne, a French theologian, mathematician and fellow Franciscan, is known for his work *Harmonie Universelle* (1637). While both men (as would be natural for men reared under Catholic influence) eschewed all metaphysical correspondences touted by their more polytheistic predecessors, they do not shy away entirely from *musica mundana*:

> Returning now to the soul’s music, we will say that it is of two kinds: heavenly and human. The heavenly sort is not only that harmony which is known to exist among things seen and known in the heavens, but is also included in the linkings of the elements and in the changing of the seasons. It is, I say, seen and known in the heavens from the revolutions, distances, and placements of the heavenly spheres, as well as from the aspects, nature, and position of the seven planets: the

> Moon, Mercury, Venus, the Sun, Mars, Jupiter, and Saturn. For it is the opinion of many ancient philosophers, notably of Pythagoras, that the revolution of so vast a machine at such speed could not possibly take place without giving forth some sound. (Zarlino, 1992, 206)

> One can prove further that the unison is more excellent than the other consonances through Astrology, which sees the consonances in the aspects of the planets. The conjunction is the most powerful and excellent of all aspects, but many deny that it merits the name of aspect, just as they deny that the unison is one of the consonances. In fact, if the conjunction of planets represents the unison —as they hold that the opposition represents the octaves, the trine the fifth, the square the fourth, and the sextile the thirds and sixths—and if the conjunction is more powerful than the other aspects, one may well say that it has a close correspondence with the unison. (Mersenne, 1992, 261)

There is an interesting anecdote, in Bob Gilmore's biography of Partch, that attests to Partch's study of these works, regarding a meeting in England with musicologist and instrument builder, Arnold Dolmetsch: "In Partch's version of the meeting, Dolmetsch, talking effusively about ancient writings on music, refers to Marin Mersenne's *Harmonie universelle* and is astonished when Partch asks him which edition of the book he is referring to, stammering out: 'For twenty years I have been talking about Mersenne, and nobody even knows who I am talking about. And now you—you ask me which edition!'" (Gilmore, 1998, 107). So while Partch appears to be primarily attuned to the practical aspects of these works, he displays a level of scholarly study that indicates exposure to esoteric Pythagorean ideas.

> Partch decried the manner in which the mathematical elegance of just intervals confounds so many, while the comparatively complex calculations of equal temperament are accepted with ease. According to Partch, this paradoxical situation resulted from the standardization of pedagogical training, the mass production of musical instruments, the economic interests of the recording and performance industries, and—perhaps most importantly—the stylistic developments in Western music itself. In each case, equal temperament's predominance has ostensibly been passed from one generation to the next without question or comment. The realization of this led Partch to suspect an ideological force behind equal temperament's thrust (Harlan, 2007, 1).

Just Intonation is a term that describes any system of musical tuning that bases its intervallic relationships on the whole-number ratios of the harmonic series, as opposed to tempered systems like our 12-TET. These ratios are used to define pitch-identities. We have no A, B, C-sharp or D-flat in Just Intonation; we begin with one frequency identified as 1/1, and every other pitch is identified by its relationship to the 1/1. Figure 2 illustrates Just Intervals, which establishes the ratio/tone-identities comprising Partch's discussion:

Fig. 2: Just Intervals and their 12-TET Equivalents

Just Interval	12-TET Equivalent	Deviation in Cents
2/1	Octave	0
3/2	Fifth	+2
4/3	Fourth	-2
5/4	Major Third	-14
6/5	Minor Third	+16
9/8	Whole Tone	+4
16/15	Semitone	+12

Using this figure, we can identify three different types of ratio/tone-identities. First, there are what Harry Partch referred to as *odentities*—ratios expressing tone-identities that are overtones of 1/1. These ratios, if expressed as fractions, must have a denominator that is a power of 2. If we examine Pythagoras' diatonic scale, we see this holds true of each tone except the 4/3 (1/1—9/8—81/64—4/3—3/2—27/16—243/128—2/1). Second are the ratios Partch called *udentities*, which are undertones of 1/1. Udentities are easily identified by having powers of 2 as numerators (4/3, 8/7, 8/5, 16/9, etc.). The third are referred to in this work as *derivative* intervals, because they only exist as tone-relationships derived from the harmonic series but not expressible as overtone or undertones of 1/1. A good example of this is the just minor third, 6/5. One way of thinking about this interval is that it is the relationship of the 6th overtone to the 5th overtone, with that relationship then applied to the 1/1.

It should be added, however, that Partch did not use these invented terms, *odentity* and *udentity*, only to refer to tones based off the nexus of 1/1. In Partch's system it is perfectly acceptable to say, for instance, "the 9 Odentity of an 8/5 Unity is 9/5." This is a way of expressing tonal relationships within the framework of a *Tonality*, which in Partch's terms was a hexachord using any tone as a base representing 1 and involving the upper partials of 3, 5, 7, 9, and 11. However, this is beyond the scope of this investigation; we shall adhere to the 1/1 as our nexus. (It should be noted that when 4/3 is used as a nexus all the numbers of Pythagoras' diatonic can be found as overtones of 4/3, if we are willing to sacrifice the simplicity of small numbers. Dean Drummond, composer/curator of the Harry Partch Instrumentarium, frequently pointed out to this author that all udentities are expressed as overtones of pitches other than 1/1 if we are willing to deal with more complex numbers, so the method of "udentifying" pitches is primarily for simplicity's sake, and to perhaps give access to interesting philosophical speculation: odentity:undentity::yang:yin, etc.)

Next, let us distinguish the different methodologies of using numbers to represent tones. These consist of: (1) ratios of string length, (2) ratios of actual frequency, and (3) ratios for the purposes of Just Intonation. For ratios of string length, we must keep in mind that the shorter the string, the higher the tone. As frequency increases, length decreases. Divisions of 1 produce overtones and multiples of 1 produce undertones, while for ratios of actual frequency the inverse is true. See comparative chart below, for the first five overtones and undertones of G. Henceforth we will use G as our 1/1 rather than C because that is what Harry Partch used, so we will not confuse the issue by doing otherwise.

Fig. 3: String Length/Frequency Comparison Chart

12-TET NOTE NAME	STRING RATIO	ACTUAL FREQUENCY RATIO
G	1/1	1/1
G(1)	1/2	2/1
G(-1)	2/1	1/2
D(1)	1/3	3/1
C(-2)	3/1	1/3
G(2)	1/4	4/1
G(-2)	4/1	1/4
↓B(2)	1/5	5/1
↑E♭(-3)	5/1	1/5

The reciprocal relationship of frequency to string length is interesting to note, because it gives us the chance to ponder this relationship for greater meaning. Hans

Kayser likened this relationship to Einstein's theory of space and time as one continuum rather than discrete concepts (string length representing objects in space, frequency representing time) (Kayser 2006, 14).

Before moving on to discuss the third method, it is necessary to clarify two aspects of translating just intervals into 12-TET pitch classification. Firstly there is the number, in parentheses, following each pitch, which represents the octave. The G without a number following it is G (just below middle C) on the piano, and G(1) would be the first octave above, while the G(-1) is the first below. Also notice in the 12-TET tone names that some pitches are preceded by the symbols (↑) and (↓). These come from Dean Drummond's method of translating just intervals into 12-TET notation. The method utilizes the standard microtonal division of 100 cents per equal-tempered semitone. Ranges are identified as plus or minus 12.5 cents, giving each possible pitch name a possible range of 25 cents to fall within. In other words, the pitch ↑G would fall somewhere between 12.5 and 37.5 cents sharper than an equal-tempered G. You can see in Figure 3 that the B of the 5th overtone of G is written ↓B, because the overtone B is 14 cents lower than an equal-tempered B. Therefore, it falls into the -12.5 to -37.5 range and is designated with the downward arrow. Also note the symbol (𝄲) is used to represent a half-sharp, or sharpness by half a semitone (50 cents), while the symbol (𝄳) represents a half-flat. The chart below (Figure 4) demonstrates this.

Fig. 4: Accidentals/Cents Comparison Chart

CENTS	PITCH
100	G♯
87.5	
75	↓G♯
62.5	
50	G𝄲
37.5	
25	↑G
12.5	
0	G
-12.5	
-25	↓G
-37.5	
-50	G𝄳
-62.5	
-75	↑F♯
-87.5	
-100	F♯

There is only one flaw in this system insofar as the pitches of Partch's Tonality Diamond are concerned. Two pairs of pitches on the Diamond (9/5 & 20/11 and 11/10 & 10/9) end up having the same 12-TET-referenced notation. 9/5 is 18 cents sharper than F, while 20/11 is 35 cents sharper; therefore both fall into the 12.5<x<37.5 range of ↑F. Similarly, 11/10 is 35 cents flatter than A, while 10/9 is 18 cents flatter, so both fall in range of ↓A. To avoid confusion we will differentiate 9/5 as +↑F and 11/10 as -↓A.

The third method of pitch classification by ratio is that which is employed in the composition of music in just intonation. In this method, pitch names are expressed as fractions that translate into numbers between 1 and 2, such as 3/2 (1.5), 5/4 (1.25), 7/4

(1.75), and so on. Just as in standard composition pitch names are not differentiated by octave, so we do not identify a pitch as a 14/4 when we already know that pitch as a 7/4 (see the Law of Octaves, below). Octave differentiation occurs naturally in notation. We do not even use 2/1, because we already know it as 1/1. So any pitch other than 1/1 has to fit into the equation 1 < x/y < 2. Therefore the above graph, once octave duplications are removed, would look like this for the purposes of just intonation:

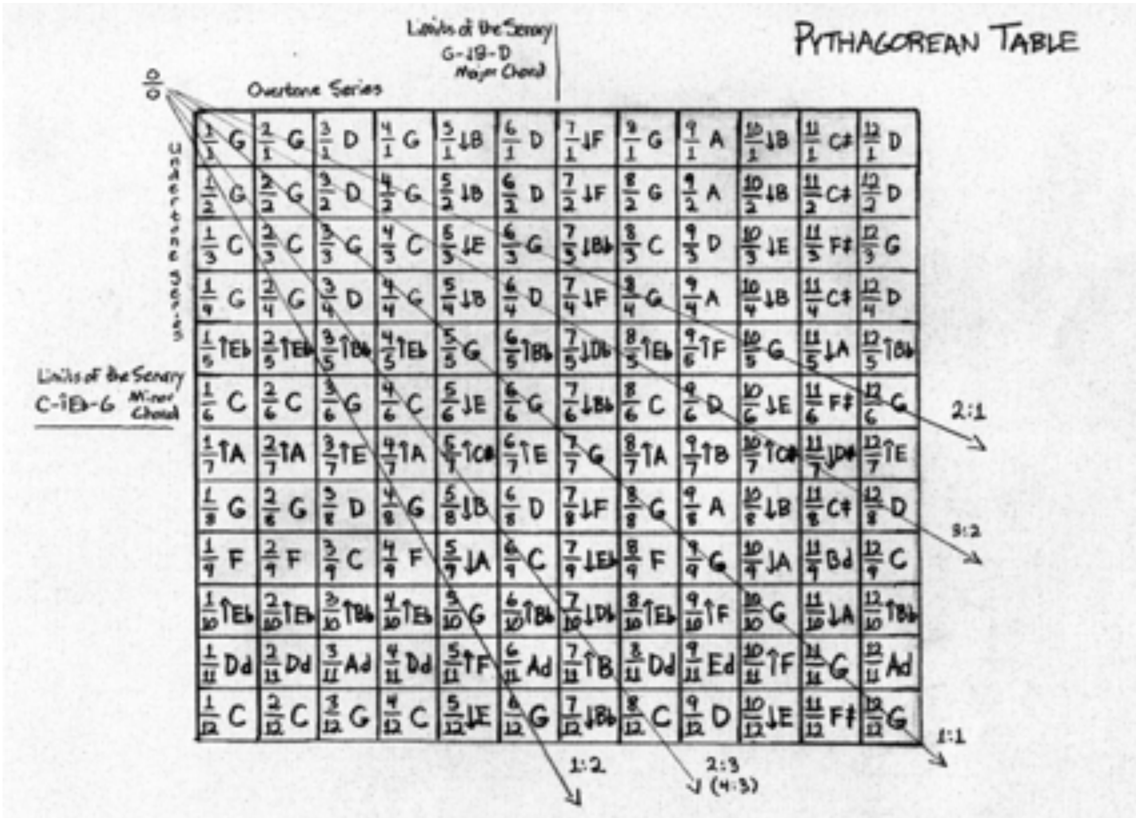
G	1:1
D	3:2
C	4:3
↓B	5:4
↑E♭	8:5

It is important to understand the role of the number 2, or the Law of Octaves. We see that our starting frequency of the note G repeats at all powers of 2: 2, 4, 8, 16, and would continue at 32, 64, 128, 256, etc. If you examine this according to frequency in Hertz you see that a doubling of vibrations produces the octave. Any note anywhere in the scale is double the vibration of the octave below, and half the vibration of the one above. This can be applied to all notes anywhere in the series. If a new note-identity occurs, for instance at the 3rd partial where D appears, then 3 multiplied by any power of 2 will also produce a D (3, 6, 12, 24, 48, etc.). Understanding this law is essential for knowing how to octave-reduce the ratios so that they fall within the 1/1-2/1 octave.

The second rule of importance is the Law of Three. The third overtone produces the note-identity of D, and the interval of what pianists would refer to as a fifth. If we examine all powers of 3 within the series, we see the Circle of Fifths, as it were. 9 is A, 27 is E, 81 is B, 243 is F♯, and so forth. Pythagoras determined a diatonic scale through use of the number 3. But this scale has some glaring flaws—the first being the use of the 81st partial for the interval of a major third. This overly large interval is often referred to as a ditone (since it is the sum of two 9/8 whole tones) and it is even sharper than an equal tempered major third, which is already sharper than the just 5/4 by 14 cents. The second flaw in the Pythagorean diatonic is that cycling through twelve 3/2's to get back to an octave of 1/1 does not work, in truth. Starting at a C and going up twelve 3/2's brings you to a C that is sharper than the one you started with, by a ratio of 531,441/524,288. This ratio is known as the Pythagorean "comma," equal to 23.46 cents. Nonetheless we have retained the concept of the Circle of Fifths in Western music, and have made the tuning fit by tempering each fifth to 3 cents flat, thereby spreading the comma over the tonal spectrum by fudging the math.

Now that these concepts are fully explained, let us take another look at the Lambdoma with the associated 12-TET tone names added for correspondence with the Partch system.

Fig. 5: Pythagorean Table with Pitch Names based on Partch's G=1/1*



(Drawing & Photo by Author)

* In accordance with what was said previously, the numbers 9/5 and 9/10 should be listed as +↑F, and the numbers 11/5 and 11/10 should be listed as -↓A.

This drawing helps clarify the process by which the Tonality Diamond is a transfiguration of the Lambdoma, conceptualized mathematically in harmonic proportion. In order to make clear how the Lambdoma becomes the Tonality Diamond we will go through a step-by-step process of conversion from one to the other. First we will revert the Lambdoma back to the original Lambda configuration (see Chapter 4, Plato), and remove all the ratios between the outer arms.

1/1	
1/2	2/1
1/3	3/1
1/4	4/1
1/5	5/1
1/6	6/1
1/7	7/1
1/8	8/1
1/9	9/1
1/10	10/1
1/11	11/1

Next we remove any unnecessary numbers by eliminating all octaves. This effectively does away with any powers of 2 or primes multiplied by powers of 2, so ratios containing the numbers 2, 4, 6, 8, 10, and 12 are eliminated. (It may appear that all we are doing is removing the even numbers, but that is not the reason for the removal, and even numbers will reappear later in the process.)

1/1

1/3 3/1

1/5 5/1

1/7 7/1

1/9 9/1

1/11 11/1

The next step is to modify all ratios other than 1/1 so that they fit within the equation $1/1 < x/y < 2/1$, thus becoming reduced to the octave between 1/1 and 2/1. This is achieved in each case by changing the number 1, in either the numerator or denominator, into the lowest power of 2 that serves the purpose. In the end, it becomes this:

1/1

4/3 3/2

8/5 5/4

8/7 7/4

16/9 9/8

16/11 11/8

Next it is necessary for the 3 and 5 identities to change places. This was done by Partch for practicality's sake in making a playable instrument; with those pitches changed around the lines become a stepwise progression of various types of thirds (12-TET lingo), so long as you are descending in pitch along the left (undertone) arm and ascending the pitch along the right (overtone) arm. If we add the pitch names this should be easy to see.

(G)

1/1

(↑E♭) 8/5 5/4 (↓B)

(C) 4/3 3/2 (D)

(↑A)8/7 7/4 (↓F)

(F) 16/9 9/8 (A)

(D𝄳) 16/11 11/8 (C𝄲)

Each arm now comprises what Partch would call an otonality and a utonality. In Partchian terminology a tonality is a set of six pitches starting at a numerary nexus (in this case, 1/1 is the nexus), and the other pitches relate to the nexus according to the overtone or undertone series in a sequence of 1-5-3-7-9-11—hence, o-tonality or u-tonality, respectively. Then the next step is accomplished by first rotating the lambda ninety degrees to the left, so that 1/1 sits in a far left marginal position, and then adding a 1/1 generator line across the middle.

11/8

9/8

7/4

3/2

5/4

1/1 1/1 1/1 1/1 1/1 1/1

8/5

4/3

8/7

16/9

16/11

After this, the other pitches are filled in. There are different ways of approaching this, and it is somewhat more complicated than filling in the Lambdoma pitches. The simplest way to think about it is to create five more ascending otonalities using the udentities as the nexuses; in other words, begin by filling in the pitches along the ascending diagonal going up from 8/5, using the first line as the model. Since you already have 8/5 and 1/1 in place as the first two pitches of an 8/5 otonality, for the next one you can find the 3 odentity of 8/5 by multiplying 8/5 by 3/2 (adding intervals is accomplished through multiplication). 8/5 x 3/2 = 24/10 = 12/5, then octave-reduced (dividing 12 by 2) to 6/5. The next number would be 8/5 x 7/4 (7/5), the next 8/5 x 9/8 (9/5), and the next 8/5 x 11/8 (11/10). Repeat the same process for each udentity in the lower arm.

It can also be approached in this manner: any ratio in the matrix is an intersection of two lines, an ascending diagonal and a descending diagonal. These twelve lines are defined by the prime number in the ratio at the outer left; meaning, for example in the 8/5 ascending otonality line, all denominators must be octaves of 5 (5, 10, 20, 40, etc.). By the same token, any ratio in the 5/4 descending utonality line must have an octave of 5 as a numerator. The end result will be this:

					11/8					
				9/8		11/10				
			7/4		9/5		11/6			
		3/2		7/5		3/2		11/7		
	5/4		6/5		7/6		9/7		11/9	
1/1		1/1		1/1		1/1		1/1		1/1
	8/5		5/3		12/7		14/9		18/11	
		4/3		10/7		4/3		14/11		
			8/7		10/9		12/11			
				16/9		20/11				
					16/11					

Hence, the Pythagorean Table becomes the Tonality Diamond. For the final basis of comparison, if we look at the Diamond with pitch names in place of the ratios and then refer back to the Table in Fig. 5, you will see that every pitch in the Table is accounted for in the Diamond, without all the repeats involved in the 12 x 12 matrix prior to octave reduction. The only repeats in the Diamond are the 3/2 (D) and 4/3 (C), which each repeat once, and the 1/1 which appears six times across the generator line.

C‡

A　-↓A

↑F　+↑F　F‡

D　↓D　D　↑D♯

↓B　↑B♭　↓B♭　↑B　B𝄳

G　G　G　G　G　G

↑E♭　↓E　↑E　↓E♭　E𝄳

C　↑C♯　C　↑B

↑A　↓A　A𝄳

F　↑F

D𝄳

So here the 12 x 12 Lambdoma matrix is compressed from 144 tone-numbers with multiple repeats across eight octaves to a more practical configuration, with 29 pitch-identities spread over 36 positions compressed into a four-octave range. This matrix was the basis of several instruments Partch invented and formed the foundation stone of his musical theory.

Understanding Philosophy: Applied Mathematics or Applied Harmonics?

As various scholars have articulated, the Lambdoma has remarkable consequences for the theory of harmony. The correlation of this linkage between "inner and outer worlds" is also inherent in the Lambdoma itself. Review the table and notice the 1/1 as repeating entity moving diagonally through the center of the matrix (1/1, 2/2, 3/3, 4/4, 5/5, 6/6...). This 1/1 generator line, as it is called, divides the world of numbers from quantities less than one, becoming progressively smaller, from quantities more than one, growing progressively larger, each potentially proceeding all the way to 1/∞ and ∞/1, respectfully. This mathematical reality easily intercepts with philosophical concepts such as the duality of man, the polarization of complementary opposites, the doctrine of yin and yang, etc.

Although the Lambdoma has had an enormous influence on instrument builders, it is even more incredible that Partch inadvertently took the concept of the Lambdoma and translated it into the Tonality Diamond, and then brought it into materiality as the Diamond Marimba, thus transmuting the abstract theory into corporeality. Partch built the instrument in 1946 in Madison, Wisconsin, with the help of Warren E. Gibson. The blocks are a mixture of Pernambuco and Brazilian rosewood, and the resonators are Brazilian bamboo. This instrument uses the precise configuration of the Tonality Diamond for its layout of pitches, and although Partch uses the 29 pitches of the 11-limit system as the basis for several other instruments he built (Kithara I & II, Bass Marimba, the core of the Quadrangularis Reversum), the Diamond Marimba is the only one that serves as an exact physical manifestation.

As Partch continued to work with the Tonality Diamond he made the decision to fill in some large intervallic gaps that occur in the 29-tone 11-limit scale with some extra pitches, eventually settling on a 43-tone scale as the basis for his system and instruments (although his compositions are not limited to those pitches). Choices he made in this regard reveal the depth of his research and understanding of Pythagorean music theory. For instance, the first pitch above 1/1 in his system is 81/80. The interval 81/80 is known as the comma of Didymus (also called the *syntonic comma, diatonic comma,* or *Ptolemaic comma*). It is the difference between a just major third (5/4) and a Pythagorean *ditone* (81/64).

Yet regardless of how much research Partch did for *Genesis of a Music*, the fact remains that it would have been virtually impossible for Partch to have known about the Lambdoma before writing it. He did not speak German, Greek, or Latin, and the first English mention of the Lambdoma in print would have been Ernst Levy and Siegmund Levarie's *Tone: a Study in Musical Acoustics*, published in 1968. So how did this seemingly uneducated man, who lived as a hobo for the better part of twelve years, manage the spectacular accomplishment of advancing Pythagorean music theory forward even beyond where von Helmholtz or von Thimus trod? Even more intriguing is the fact that Partch invented a Lambdoma instrument (for the Diamond Marimba certainly is a Lambdoma instrument in spite of Partch's supposed ignorance of the original Pythagorean concept), with the intention of forcibly altering consciousness and perception, thereby restoring mathematical theory to the posture of First Principle in universal harmony.

This study has aimed to open some profound questions regarding the relevancy and efficacy of our conventional wisdom regarding the philosophers of antiquity and the widely held theories of musical harmony, by way of compiling the history of an obscure branch of ancient knowledge that has reemerged after a long sleep. We call it Pythagorean, in spite of uncertainty as to its true origins, but the name serves to classify it as belonging to a larger framework of endeavor and lineage of great minds. The Lambdoma is many things to many people—a simple mathematical construct, a source for the theory of music, a grand expression of universal principle, the basis for a theology of numbers, a cosmic truth for ongoing contemplation.

This also suggests that Platonic ideals cannot not be fully understood without knowledge of harmonics and thereby ancient Greek philosophy is rather an applied music theory. Since much of what Plato taught supported the concept of *musica universalis* as the basis for understanding the world, and since he went so far as to declare in *Republic* that music and gymnastics are the only important subjects to train children in until they reach adulthood (Jowett 1966,105-10), it would follow that the mathematical allegories in *Timaeus*—which are easily explained through harmonics and remain quite a puzzle otherwise—are certainly intended by Plato to be understood through realization of the numbers as tones. Plato's metaphors demand an experience of these numbers for their greater meaning to have impact on the mind of the reader. Without that meaning, numbers are mere quantities. To have the *musical experience* of these metaphors, one must understand harmonics. Therefore Platonic scholarship can hardly be complete without that understanding. Ernest C. McClain, a colleague of Ernst Levy and Siegmund Levarie, explored this topic in great depth in his book *The Pythagorean Plato: Prelude to the Song Itself* (1978).

To the greater question of whether music theory can be complete without understanding *musica universalis* and the greater significance of harmonics, I would argue this. There is no doubt that musicians can be good at their craft without even the smallest amount of musical education; some of the most impressive musician/composers of the twentieth century were largely self-taught—Harry Partch being one of them. But many of these same composers, even if they are vehemently anti-religion, claim a sense of music connecting them to the cosmos in a manner that could be characterized as mystical or spiritual—even if they had only heard of Pythagoras in their high school geometry class. Partch may have had no use for religion, but he did speak frequently of the idea of music as magic in the truest sense—and all sages, shamans, magi, wizards, etc. throughout all history and cultures would wholeheartedly agree. For Pythagoras and those that followed him, the discovery of the tone-number relationship was precisely what put the magic nature of music beyond the shadow of doubt. Not only could one perceive that magic with the senses, but the structure of reality was rendered knowable through the study of the monochord. The mind of man could unify with the mind of the Creator. We were no longer adrift in the vagueness of intuition; tuning could be calculated to mirror the perfect numbers of the divine construction.

From this majestic starting point, Western music moved through centuries of change whereby its roots were forgotten, and means of expression shifted—from modalism to major-minor tonality to the harmonic experimentation of the twentieth century. Another great composer, Arnold Schoenberg, used the concept of the twelve by twelve matrix to do away with all fixed rules of tonality and "emancipate the dissonance," creating a style of composition that sought to end the reign of the 1/1 as the point from which all harmonic relationship emanated. The conventional rules had become synonymous with tyranny and oppression, and music could only be redefined for the post-WWII world by shattering all our preconceptions about its nature.

Of course one cannot shatter what is absolute and unchanging, and the mistake in this case was to confuse the arbitrary rules decided on by unenlightened men (who sought to make music conform to systems of belief) with the laws of the universe which only nature can reveal. This is the place to which we can return, if we so choose. What can we wrest from the Lambdoma, using our capacity for poetry and metaphor to tease meaning from these numbers that Pythagoras revered as gods? The Lambdoma can be extended to include every conceivable rational number, and yet the presence of the 1/1 will be eternally present down through the middle all the way to the final corner. If the ratios are given to represent all things that exist, the neutral ground of the 1/1 generator line divides all creation into numbers either greater or lesser than the 1/1, revealing the duality inherent in the world between positive and negative, masculine and feminine, ascending and descending, and all other aspects of reality eternally subject to the laws of polarity.

It is also worth noting that each tone-number-identity displayed here in its aspect as a ratio is a relationship, a bridge between two discrete archetypes. What does this tell us? What does it say to define Unity not as simply the One, but as 1/1, the relationship of Unity to Itself? Perhaps it calls to mind the dictum of the Oracle at Delphi—"Know Thyself"—that was once held as the only true path to understanding the universe and our role within it? Perhaps it tells us that relationship is the basis of all harmony, and that tones only acquire meaning by their relationship to the 1/1, and otherwise will only exist as arbitrary frequencies without greater purpose. At any rate, an increased understanding of mathematical principles in the field of music is not only useful in application but also potentially crucial to students of philosophy as well.

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